Tutorial Note VIII

Exercise 0.1

Suppose that $u \in C^2(\mathbb{R}^2)$ is a solution to the 1D wave equation. Then

$$u(x+h.t+k)+u(x-h,t-k)=u(x+k,t+h)+u(x-k,t-h)$$

for all x, t, h, $k \in \mathbb{R}$.

Remark 0.1

The parallelograms with vertices $P_1=(x+h,x+k)$, $P_2=(x+k,x+h)$, $P_3=(x-h,x-k)$, $P_4=(x-k,x-h)$ are usually called characteristic parallelograms.

Proof. We integrate $u_{tt}-u_{xx}$ on the characteristic parallelogram P, and by the Green formula we have

$$0=\int_{L_1+L_2+L_3+L_4}(u_x\,\mathrm{d}t+u_t\,\mathrm{d}x),$$

where $L_1=[P_1\to P_2]$, $L_2=[P_2\to P_3]$, $L_3=[P_3\to P_4]$, $L_4=[P_4\to P_1]$. Since

$$\int_{L_1}(u_x\,\mathrm{d}t+u_t\,\mathrm{d}x)=u(P_1)-u(P_2);$$
$$\int_{L_2}(u_x\,\mathrm{d}t+u_t\,\mathrm{d}x)=u(P_3)-u(P_2);$$
$$\int_{L_3}(u_x\,\mathrm{d}t+u_t\,\mathrm{d}x)=u(P_3)-u(P_4);$$
$$\int_{L_4}(u_x\,\mathrm{d}t+u_t\,\mathrm{d}x)=u(P_1)-u(P_4),$$

the conclusion follows immediately. □

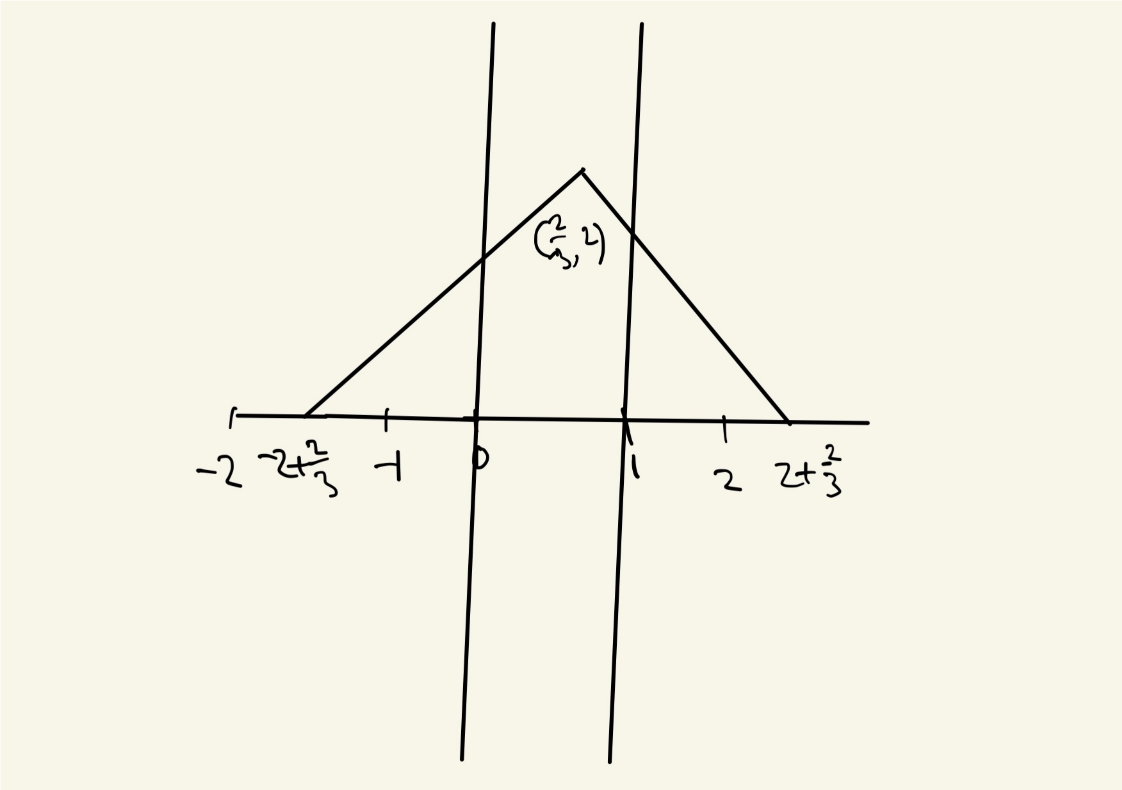
Exercise 0.2

Suppose that u is a solution to the IBVP:

$$\begin{cases} u_{tt}-u_{xx}=0 & (x,t)\in(0,1)\times\mathbb{R}^+; \\ u(0,t)=0,\ u(1,t)=0 & t\in\mathbb{R}+; \\ u(x,0)=x^2(1-x),\ u_t(x,0)=(1-x)^2 & x\in[0,1]. \end{cases}$$

Find the value of $u(2/3,2)$.

Proof.



Similar to heat equations, we also solve it by odd extensions. However, thanks to finite speed of propagation, unlike heat equations, it could be solved more easily. Suppose that $\widetilde{\varphi}$ and $\widetilde{\psi}$ are odd extensions of $x^2(1-x)$ and $(1-x)^2$ across $(0,0)$ and $(1,0)$. Then

$$u\left(\frac{2}{3}, 2\right) = \frac{\widetilde{\varphi}(-2+2/3) + \widetilde{\varphi}(2+2/3)}{2} + \frac{1}{2}\int_{-2+2/3}^{2+2/3} \widetilde{\psi}(y)\,\mathrm{d}y.$$

Since the period of $\widetilde{\varphi}$ is 2 and there are cancellations in the integral, it is easy to see that

$$u\left(\frac{2}{3}, 2\right) = \widetilde{\varphi}\left(\frac{2}{3}\right) = \frac{4}{27}.$$

□

Exercise 0.3

Solve the following IBVP:

$$\begin{cases} u_{tt} - u_{xx} = 0 & (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+; \\ u(0,t) = 0 & t \in \mathbb{R}^+; \\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x) & x \in [0,\infty), \end{cases}$$

where $\varphi(0) = 0$, $\varphi''(0) = 0$, $\psi(0) = 0$.

Proof. Instead of using odd extensions, we use the method of characteristics to solve it. First, let $v = u_x + u_t$ and we solve v. v satisfies $v_t - v_x = 0$. For $(x_0, t_0) \in \mathbb{R}^+ \times \mathbb{R}^+$, let $f(s) = v(x_0 + s, t_0 - s)$, then $f'(s) = 0$. Since $f(t_0) = \varphi'(x_0 + t_0) + \psi(x_0 + t_0)$, $v(x_0, t_0) =$

$f(0) = \varphi'(x_0 + t_0) + \psi(x_0 + t_0)$. Then we solve u. We divide it into two cases because the characteristics $(s, s+c)$ hit x-axis or y-axis. For $(x_0, t_0) \in \mathbb{R}^+ \times \mathbb{R}^+$, let $g(s) = u(x_0+s, t_0+s)$, then $g'(s) = \varphi'(x_0+t_0+2s)+\psi(x_0+t_0+2s)$. If $(x_0, t_0) \in \{t \leq x\}$, then $g(-t_0) = \varphi(x_0-t_0)$ and

$$\begin{aligned} u(x_0, t_0) = g(0) &= \varphi(x_0 - t_0) + \int_{-t_0}^{0} (\varphi'(x_0 + t_0 + 2s) + \psi(x_0 + t_0 + 2s))\,\mathrm{d}s \\ &= \frac{\varphi(x_0 - t_0) + \varphi(x_0 + t_0)}{2} + \frac{1}{2}\int_{x_0-t_0}^{x_0+t_0} \psi(y)\,\mathrm{d}y. \end{aligned}$$

If $(x_0, t_0) \in \{t > x\}$, then $g(-x_0) = 0$ and

$$\begin{aligned} u(x_0, t_0) = g(0) &= \int_{-x_0}^{0} (\varphi'(x_0 + t_0 + 2s) + \psi(x_0 + t_0 + 2s))\,\mathrm{d}s \\ &= \frac{\varphi(t_0 + x_0) - \varphi(t_0 - x_0)}{2} + \frac{1}{2}\int_{t_0-x_0}^{t_0+x_0} \psi(y)\,\mathrm{d}y. \end{aligned}$$

And it is easy to check that the above u is a solution. □

Remark 0.2

By the method of characteristics, you may try to solve the Goursat problem:

$$\begin{cases} u_{tt} - u_{xx} = 0 & -t < x < t; \\ u(t,t) = \varphi(t) & t \geq 0; \\ u(-t,t) = \psi(t) & t \geq 0, \end{cases}$$

where $\varphi(0) = \psi(0)$.